JOURNAL OF APPROXIMATION THEORY **38**, 221–239 (1983)

Polygonal Approximation of Plane Convex Compacta*

PETAR S. KENDEROV

Institute of Mathematics, Bulgarian Academy of Sciences, 1090 Sofia, Bulgaria

Communicated by G. G. Lorentz

Received November 20, 1981

1. INTRODUCTION

Let R^2 be the usual two-dimensional plane with the Eucledean norm $|\cdot|$. By CONV we denote the set of all convex compact subsets of R^2. The Hausdorff distance between two elements A_1, A_2 of CONV is given by $h(A_1, A_2) = \inf\{t > 0\colon\ A_1 \subset A_2 + tB,\ A_2 \subset A_1 + tB\}$, where $B = \{P \in R^2\colon |P| \leqslant 1\}$ is the unit circle, $C_1 + C_2 = \{P_1 + P_2\colon\ P_i \in C_i,\ i = 1, 2\}$ is the Minkowski sum of C_1, C_2 from CONV and $tB = \{tP\colon P \in B\}$. For every integer $n \geqslant 3$ we denote by POLY_n the set of all convex polygons with not more than n vertices. The elements of POLY_n will be called n-gons. The n-gon Δ_0 is said to be a best Hausdorff approximation in POLY_n for the set $A \in \mathrm{CONV}$ if $\inf\{h(A, \Delta)\colon \Delta \in \mathrm{POLY}_n\} = h(A, \Delta_0)$. The existence of at least one best Hausdorff approximation for any $A \in \mathrm{CONV}$ follows from the well-known Blaschke "selection theorem" asserting that every bounded sequence of n-gons (n fixed) contains a subsequence converging in the Hausdorff metric to some n-gon. In general, as examples like the unit circle or the unit square show, the best approximation is not unique. Nevertheless the "majority" of the elements of CONV have unique best approximation in any POLY_n, $n \geqslant 3$. The "majority" here means: with an exception of some first Baire category subset of the locally compact metric space (CONV, h), all convex compact subsets of R^2 have unique best approximation in POLY_n for every $n \geqslant 3$ (Theorem 3.5). To prove this we give (and use) a necessary condition for $\Delta \in \mathrm{POLY}_n$ to be a best approximation for $A \in \mathrm{CONV}$. This condition (Theorem 2.1) coincides with the classical alternating condition in

* A part of this work was done while the author was at the University of Frankfurt am Main (Federal Republic of Germany) as a research fellow of the Alexander von Humboldt Foundation.

0021-9045/83 $3.00

the problem of uniform Čebyšhev approximation by polynomials. But, in the present situation, it is very far from being a sufficient condition.

If we impose on the best approximating n-gon the additional requirement of having one of its sides perpendicular to a given vector, directed "outward Δ" (i.e., we consider another approximation problem), then the alternating property completely determines a unique best approximation (Theorem 4.12).

Some of the results presented here were announced in [11] and reported at the conference "Constructive Function Theory" held in June 1981, near Varna, Bulgaria.

The way in which the sequence $\{r_n(A)\}_{n\geqslant 3}$, where $r_n(A) = \min\{h(A, \Delta): \Delta \in \mathrm{POLY}_n\}$ tends to 0 was studied by Tóth [18] Popov [15], and McClure and Vitale [1]. It is an open problem to find necessary and sufficient conditions for a given n-gon Δ to be a best Hausdorff approximation in POLY_n for some $A \in \mathrm{CONV}$. Also unknown is the answer to the following question of Sendov and Popov: Is it true that among all elements of CONV with perimeter 1, the equilateral $(n+1)$-gon (with the same perimeter) is the worst one to be approximated by n-gons? There is a result of Ivanov [6] concerning approximation by inscribed n-gons which is in favour of the "yes" answer of this question: among all $(n+1)$-gons with perimeter 1 the equilateral $(n+1)$-gon is the worst to be approximated by inscribed n-gons. The result from [12] is also in support of the positive answer.

2. Notations and Preliminary Results

Let us agree to denote the usual inner product of two points (vectors) $P_1, P_2 \in R^2$ by $\langle P_1, P_2\rangle$. Set $|P| = \sqrt{\langle P, P\rangle}$. The function defined in R^2 by the formula $s_A(P) = \max\{\langle P, X\rangle: X \in A\}$, where $A \in \mathrm{CONV}$, is called a support function of the set A. This function is positively homogeneous and is, therefore, completely determined by its values at the points of the set $S = \{e \in R^2: |e| = 1\}$. Then s_A is convex and continuous. In this way a mapping $A \mapsto s_A$ is defined from CONV into the space $C(S)$ of all continuous functions on S. Evidently $s_{A_1+A_2} = s_{A_1} + s_{A_2}$ and $s_{tA} = ts_A$ whenever $t \geqslant 0$ and $A, A_1, A_2 \in \mathrm{CONV}$. Since any point which does not belong to a given compact convex subset of the plane can be strictly separated from it by a hyperplane, we can prove that for any $A_1, A_2 \in \mathrm{CONV}$ the relation $A_1 \subset A_2$ is equivalent to the assertion $s_{A_1}(e) \leqslant s_{A_2}(e)$ for every $e \in S$. Having in mind all this and the fact that the supporting function of the unit circle $B = \{P \in R^2: |P| \leqslant 1\}$ is just the constant 1, the Hausdorff distance between two sets $A_1, A_2 \in \mathrm{CONV}$ can be expressed in the following way: $h(A_1, A_2) = \inf\{t > 0: s_{A_1}(e) \leqslant s_{A_2}(e) + t,\ s_{A_2}(e) \leqslant s_{A_1}(e) + t$ for every $e \in S\} = \inf\{t > 0: |s_{A_1}(e) - s_{A_2}(e)| \leqslant t$ for every $e \in S\} = \max\{|s_{A_1}(e) - s_{A_2}(e)|: e \in S\}$. Because of this fact the problem of approximating the elements of CONV by elements of POLY_n, $n \geqslant 3$, with respect to the Hausdorff metric is equivalent to the uniform approximation in $C(S)$ of the support functions of elements of CONV by the support functions of n-gons. In what follows we identify CONV and POLY_n with their images in $C(S)$ under the above defined mapping.

Let us accept the counterclockwise direction on S as positive. For $e_1, e_2 \in S$ we denote by $[e_1, e_2]$ the arc on S with end points e_1 and e_2 which connects e_1 and e_2 "in the counterclockwise direction." Thus $[e_2, e_1] = S \backslash (e_1, e_2)$. As with segments, by (e_1, e_2) we denote the "open" arc, i.e., $[e_1, e_2]$ without the end points e_1 and e_2. It is clear what $[e_1, e_2)$ and $(e_1, e_2]$ mean. When there is no danger of ambiguity the symbol $(e_1, e_2]$ (or $[e_1, e_2)$, (e_1, e_2), $(e_1, e_2]$) will denote the length of the corresponding arc as well.

Now we need a more precise definition of the notion n-gon.

2.0. Definition. The convex set $\Delta \subset R^2$ will be called a nondegenerated k-gon, where k is an integer, $k \geqslant 3$, if there exist points $\{P_i\}_{i=1}^k \subset \Delta$ and vectors $\{e_i\}_{i=1}^k \subset S$ such that

(1) $0 < (e_i, e_{i+1}) < \pi \; (e_{k+1} := e_1)$;

(2) $e_i \in (e_{i-1}, e_{i+1}) \; (e_0 := e_k)$;

(3) $P_i \neq P_j$ when $i \neq j$;

(4) $s_\Delta(e) = \langle e, P_i \rangle$ whenever $e \in [e_{i-1}, e_i]$.

The points P_i, $i = 1, 2, \ldots, k$ are called vertices of Δ. The segments $\overline{P_{i-1}, P_i}$ will be called sides of Δ and the vector e_i, $i = 1, 2, \ldots, k$ will be called a "side direction" of the side $\overline{P_{i-1}, P_i}$. The convex set Δ is said to be an n-gon, $n \geqslant 3$, if it is a nondegenerated k-gon for some k, $3 \leqslant k \leqslant n$.

It is easy to see (using separation argument) that every nondegenerated k-gon Δ is the convex hull of its vertices. Of course, it may be proved that the convex hull of any n points $P_1, P_2, \ldots, P_n$, $n \geqslant 3$, which are different and do not lie on one straight line, is an n-gon in the sense of the above definition. Having this in mind, we can express the Hausdorff distance between the convex set A and some (nondegenerated) n-gon in the following way: $h(A, \Delta) = \| s_A - s_\Delta \|_{C(S)} := \max\{ |s_A(e) - s_\Delta(e)|; \quad e \in S\} = \max\{\max\{|s_A(e) - \langle e, P_{i+1} \rangle|: e \in [e_i, e_{i+1}]\}; \; i = 1, 2, \ldots, n\}$. Therefore, in order to study the best approximation of A by n-gons, we have to investigate the behaviour of the function $s_A(e) - \langle e, P_{i+1} \rangle$ in $[e_i, e_{i+1}]$. This behaviour is described in the following result: Let $M \in R^2 \backslash A$, $A \in \mathrm{CONV}$. Put $d(M, A) = \min\{|X - M|: X \in A\}$. By the strict convexity of the Eucledian norm $|\cdot|$ there exists just one point $N \in A$ such that $|M - N| = d(M, A)$. Put $e^* = (M - N)/d(M, A)$. Clearly $e^* \in S$. Moreover $s_A(e^*) = \langle e^*, N \rangle$ and $d(M, A) = \langle e^*, M \rangle - s_A(e^*)$

(later we will see that this condition completely determines e^*). For the sake of simplicity we assume from now on that A has interior: $\operatorname{int} A \neq \varnothing$.

2.1. PROPOSITION. *There exists a unique vector $e' \in (e^*, -e^*)$ such that*

(a) $s_A(e') - \langle e', M \rangle = d(M, A)$.

(b) *When e runs from e^* to e' in the positive direction the function $s_A(e) - \langle e, M \rangle$ strictly increases from $-d(M, A)$ (for $e = e^*$) to $d(M, A)$ (for $e = e'$).*

(c) *For $e \in (e', -e^*]$, $s_A(e) - \langle e, M \rangle > d(M, A)$.*

Analogously, there exists a uniquely determined vector $e'' \in (-e^*, e^*)$ such that $s_A(e) - \langle e, M \rangle$ strictly increases from $-d(M, A)$ (for $e = e^*$) to $d(M, A)$ (for $e = e''$) when e runs in the negative direction from e^* to e''. For $e \in [-e^*, e'')$, $s_A(e) - \langle e, M \rangle > d(M, A)$.

To prove this we will use some elementary facts which are listed below.

2.2. LEMMA. *Let $P \neq 0$ be a point from R^2 and let $0 < [e_1, e_2] < \pi$, where $e_i \in S$, $i = 1, 2$. If $\langle e_i, P \rangle \leqslant 0$, $i = 1, 2$, then at least one of these two inequalities is strict and, for every $e \in (e_1, e_2)$, $\langle e, P \rangle < 0$.*

2.3. LEMMA. *Let the origin 0 of R^2 not belong to $A \in \mathrm{CONV}$ and let, for some $e_1, e_2 \in S$, $0 < (e_1, e_2) < \pi$, the support function of A satisfies $s_A(e_1) = s_A(e_2) = 0$. Then*

(1) *$s_A(e) > 0$ for all $e \in S \backslash [e_1, e_2]$,*

(2) *$s_A(e) < 0$ for all $e \in (e_1, e_2)$.*

Although the meaning of Lemma 2.2 is obvious, we give here a formal proof. It illustrates the elementary technics used in the sequel.

Proof. (1) It is enough to prove the inequality for all e from $(-e_1, e_1)$ and from $(e_2, -e_2)$. Denote for this purpose by Q_i, $i = 1, 2$, some points in A for which $\langle e_i, Q_i \rangle = s_A(e_i) = 0$. Then $0 = s_A(e_1) = \langle e_1, Q_1 \rangle \geqslant \langle e_1, Q_2 \rangle$ and $0 = s_A(e_2) = \langle e_2, Q_2 \rangle \geqslant \langle e_2, Q_1 \rangle$. Therefore $0 = \langle e_1, Q_1 \rangle$ and $0 \geqslant \langle e_2, Q_1 \rangle$. By Lemma 2.2 $\langle e, Q_1 \rangle < 0$ for every $e \in (e_1, e_2]$. Therefore $\langle e, Q_1 \rangle > 0$ for each e from $(-e_1, e_1)$. Then, for $e \in (-e_1, e_1)$, $s_A(e) \geqslant \langle e, Q_1 \rangle > 0$. Analogously, from $\langle e_2, Q_2 \rangle = 0$, $\langle e_1, Q_2 \rangle \leqslant 0$ and Lemma 2.2 it follows that $s_A(e) \geqslant \langle e, Q_2 \rangle > 0$ for every $e \in (e_2, -e_2)$; (1) is proved. To prove (2) we show first that $s_A(e) \neq 0$ for every e from the arc (e_1, e_2). Indeed, if $s_A(e') = 0$ for some $e' \in (e_1, e_2)$, then by the proof of (1) (applied for (e_1, e')) we would get $s_A(e_2) > 0$ which is a contradiction. Since $s_A(e)$ is a continuous function the same argument shows that $s_A(\cdot)$ must have one and the same sign on (e_1, e_2). On the other hand, by the fact that 0 does not belong to the convex set A, there must exist some $e_0 \in S$ for which $s_A(e_0) < 0$ (otherwise, $s_A(e) \geqslant 0$ for all $e \in S$ and this implies $0 \in A$). By (1) we see that e_0 has to belong to (e_1, e_2). Lemma 2.3 is proved.

Proposition 2.1 will follow from the next fact.

2.4. LEMMA. *Let A, M, N, e^*, and $d(M, A)$ be as defined just before Proposition 2.1. For every real number d, $|d| \leqslant d(M, A)$,*

$$s_A(e) - \langle e, M\rangle = d \qquad (*)$$

has just one solution e in the arc $[e^*, -e^*]$.

Proof. For brevity set $d_0 := d(M, A)$ and $s_A(\cdot) = s(\cdot)$. Since A has interior points $s(e) + s(-e) > 0$ for every $e \in S$. The existence of a solution to $(*)$ follows from the continuity of the function $s(\cdot) - \langle \cdot, M\rangle$ and the inequalities

$$\begin{aligned} s(e^*) - \langle e^*, M\rangle &= \langle e^*, N - M\rangle = -d_0, \\ s(-e^*) - \langle -e^*, M\rangle &= s(-e^*) + \langle e^*, M\rangle \\ &= s(-e^*) + s(e^*) + d_0 > d_0. \end{aligned}$$

To prove the uniqueness of the solution we consider three cases: (a) $d = -d_0$, (b) $-d_0 < d \leqslant 0$, (c) $0 < d \leqslant d_0$.

(a) We take some solution to Eq. $(*)$, i.e., $s(e_0) - \langle e_0, M\rangle = d = -d_0$, and show that $e_0 = e^*$. Indeed, $|M - N| = d_0 = \langle e_0, M\rangle - s(e_0) \leqslant \langle e_0, M - N\rangle \leqslant |M - N|$. Having in mind that $e^* = (M - N)/|M - N|$, we get from here $\langle e_0, e^*\rangle = 1$. As both vectors e_0, e^* belong to S this implies $e_0 = e^*$.

(b) Let $-d_0 < d \leqslant 0$ and suppose $s(e) - \langle e, M\rangle = d$ for some $e = e_1$ and $e = e_2$, $e_1 \neq e_2$, $[e_1, e_2] \subset (e^*, -e^*)$. Consider the set $A + (-d)B - M$, where $B = \{X \in R^2\colon\ |X| \leqslant 1\}$. This set does not contain the origin 0 of R^2 (otherwise, M would belong to $A + (-d)B$ which, in turn, implies $d_0 = d(M, A) \leqslant -d < d_0$). The support function of this set is $s(e) - d - \langle e, M\rangle$ and we have $s(e_i) - \langle e_i, M\rangle - d = 0$, $i = 1, 2$. Since $e^* \in S\backslash[e_1, e_2]$ we obtain from Lemma 2.3 the contradiction $0 < s(e^*) - d - \langle e^*, M\rangle = -d_0 - d < 0$.

(c) We can introduce a coordinate system in R^2 in such a way that M is the origin $(0, 0)$ and e^* is the vector $(1, 0)$. Let there be two vectors $e_1 \neq e_2$, $[e_1, e_2] \subset (e^*, -e^*)$ such that $s(e_i) = d$, $i = 1, 2$. Denote by $Q_i = (x_i, y_i)$, $i = 1, 2$, two points in A for which $s(e_i) = \langle e_i, Q_i\rangle$, $i = 1, 2$. Set $e(t) = (\cos t, \sin t)$, where t is a real number. Evidently $e^* = e(0)$ and $-e^* = e(\pi)$. The vectors e_i, $i = 1, 2$, can be represented as $e_i = e(t_i) = (\cos t_i, \sin t_i)$, where $0 < t_1 < t_2 < \pi$. From $d = s(e_1) = \langle e_1, Q_1\rangle = x_1 \cos t_1 + y_1 \sin t_1$ we find $y_1 =$

$(d - x_1 \cos t_1)/\sin t_1$. On the other hand, we have $d = s(e_2) = \langle e_2, Q_2 \rangle \geqslant \langle e_2, Q_1 \rangle = x_1 \cos t_2 + y_1 \sin t_2 = x_1 \cos t_2 + \sin t_2 (d - x_1 \cos t_1)/\sin t_1 = (1/\sin t_1)(d \sin t_2 - x_1 \sin(t_2 - t_1))$. Therefore $d(\sin t_1 - \sin t_2) \geqslant -x_1 \sin(t_2 - t_1)$. Since $-d_0 = s(e^*) \geqslant \langle e^*, Q_1 \rangle = x_1$ we get $-x_1 \geqslant d_0 \geqslant d > 0$. Therefore $\sin t_1 - \sin t_2 - \sin(t_2 - t_1) \geqslant 0$. But this inequality leads to a contradiction:

$$\begin{aligned} 0 &\leqslant \sin t_1 - \sin t_2 - \sin(t_2 - t_1) \\ &= 2 \sin(t_1 - t_2)/2 \cos(t_1 + t_2)/2 - 2 \sin(t_2 - t_1)/2 \cos(t_2 - t_1)/2 \\ &= -(2 \sin(t_2 - t_1)/2)((\cos(t_1 + t_2)/2) + (\cos(t_2 - t_1)/2) \\ &= -4 \sin(t_2 - t_1)/2 \cos t_2/2 \cos t_1/2 < 0. \end{aligned}$$

Proof of Proposition 2.1. By Lemma 2.4 it follows that the continuous function $s(e) - \langle e, M \rangle$ takes in $[e^*, e']$ all values between $-d_0$ and d_0 only once. Therefore it is strictly increasing when e runs from e^* to e' in the positive direction. By the same lemma the values this function takes in $(e', -e^*]$ must be bigger than d_0. The situation, where e runs from e^* to e'' in the negative direction on S, is analogous.

2.5. COROLLARY. *$d(M, A) = \max\{\langle e, M \rangle - s(e)$: $e \in [e'', e']\} = \max\{|s_A(e) - \langle e, M \rangle|$: $e \in [e'', e']\}$, $s_A(e) - \langle e, M \rangle > d(M, A)$ for $e \in (e', e'')$. There exists just one $e^* \in S$ for which $\langle e^*, M \rangle - s(e^*) = d(M, A)$.*

We need further an operation which plays an important role in our considerations. To each pair $e'', e' \in S$, $0 < (e'', e') < \pi$ and $A \in$ CONV we assign a point $M = M(A; e'', e')$, a number $d_0 = d_0(A; e'', e') \geqslant 0$ and a vector $e^* = e^*(A; e'', e') \in (e'', e')$ so that

(i) $s_A(e'') - \langle e'', M \rangle = d_0$,

(ii) $s_A(e') - \langle e', M \rangle = d_0$,

(iii) $\langle e^*, M \rangle - s_A(e^*) = d(M, A) = d_0$,

(iv) $\max\{|s_A(e) - \langle e, M \rangle| : e \in [e'', e']\} = d_0$.

Consider the lines $L' = \{X \in R^2$: $s_A(e') = \langle e, X \rangle\}$ and $L'' = \{X \in R^2$: $s_A(e'') = \langle e'', X \rangle\}$. Since $0 < (e'', e') < \pi$ there exists only one intersection point $\bar{M}$, i.e., $\langle \bar{M}, e' \rangle = s_A(e')$, $\langle \bar{M}, e'' \rangle = s_A(e'')$ (see Fig. 1). There are two possibilities: (a) $\bar{M} \in A$, (b) $\bar{M} \notin A$. In case (a) we put $d_0 = 0$, $M = \bar{M}$ and take e^* arbitrarily in (e'', e'). All requirements are fulfilled because of the following simple fact:

2.6. LEMMA. *Let $A \in$ CONV and $s(e)$ be its support function. Let $e_i \in S$, $i = 1, 2$, and $0 < (e_1, e_2) < \pi$. If for some $M \in A$ $s(e_i) = \langle e_i, M \rangle$, $i = 1, 2$, then, for every $e \in (e_1, e_2)$, $s(e) = \langle e, M \rangle$ and $s(e) > \langle e, X \rangle$ for $X \in A$, $X \neq M$.*

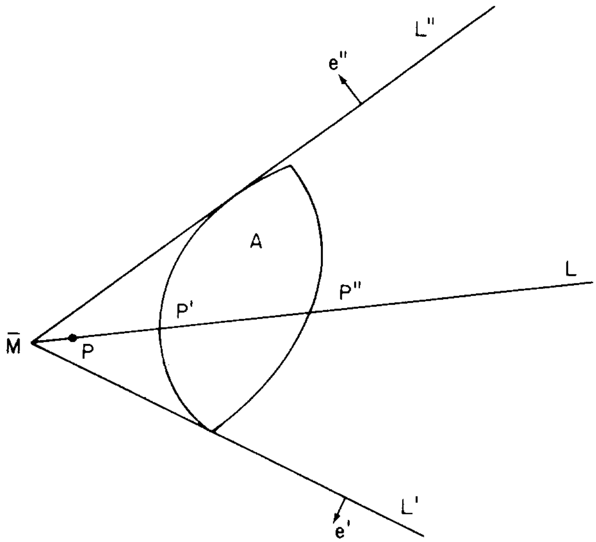

FIGURE 1

Proof. Take some $e_3 \in (e_1, e_2)$ and let $N \in A$ be a point for which $\langle e_3, N\rangle = s(e_3)$. Then $\langle e_i, N - M\rangle \leqslant 0$ $i = 1, 2$. If $N \neq M$, by Lemma 2.2 it follows that $\langle e, N - M\rangle < 0$ for $e \in (e_1, e_2)$. On the other hand, $\langle e_3, N\rangle \geqslant \langle e_3, M\rangle$ and therefore $0 > \langle e_3, N - M\rangle \geqslant 0$, which is a contradiction.

(b) $\bar{M} \notin A$. Then $d(\bar{M}, A) > 0$. Consider the bisector line $L = \{X \in R^2$: $\langle e' - e'', X - \bar{M}\rangle = 0\}$ passing through $\bar{M}$ and intersecting the set A (Fig. 1) in the segment $[P', P'']$. When a point P on L moves from $\bar{M}$ towards P' the function $d(P, A)$ decreases from $d(\bar{M}, A) > 0$ to $0 = d(P', A)$. At the same time the function $f(P) = s_A(e') - \langle e', P\rangle = \langle e', \bar{M}\rangle - \langle e', P\rangle = \langle e'', \bar{M}\rangle - \langle e'', P\rangle = s_A(e'') - \langle e'', P\rangle$ increases from $0 = f(\bar{M})$ to $f(P') > 0$. Hence, on the line L, there exists just one point M between $\bar{M}$ and P' such that $f(M) = d(M, A)$. This point M and $d_0 := d(M, A)$ satisfy (i) and (ii). Proposition 2.1 and Corollary 2.5 imply that (iii) and (iv) are also fulfilled.

The next result reveals one important extremal property of this construction. It shows that in the arc $[e'', e']$ the function $\langle e, M\rangle$ approximates $s_A(e)$ better than any other function of the type $\langle e, P\rangle$.

2.7. PROPOSITION. *Let* $A \in \text{CONV}$, $P \in R^2$ *and* $e_1, e_2 \in S$, $0 < (e_1, e_2) < \pi$. *Set* $d_i = s_A(e_i) - \langle e_i, P\rangle$, $i = 1, 2$, *and* $d_3 = \max\{\langle e, P\rangle - s_A(e)$: $e \in [e_1, e_2]\}$. *If for some pair of unit vectors* e'', e', $0 < [e'', e'] < \pi$, we have

(v) $[e'', e'] \subset [e_1, e_2]$,

(vi) $d_0 := d_0(A; e'', e') \geqslant \max\{d_i\colon\ i = 1, 2, 3\}$, *then* $d_0 = d_1 = d_2 = d_3$ *and* $P = M(A; e'', e'')$. *Moreover, if* $d_0 > 0$, *then (in addition)* $e'' = e_1$, $e' = e_2$.

Proof. Let us first consider the case when $d_0 = 0$. That is, the point $M = M(A; e'', e')$ satisfying the conditions $\langle e'', M\rangle = s_A(e'')$, $\langle e', M\rangle = s_A(e')$ belongs to A. In this case M lies on L' and L''. Since $d_3 \geqslant -d_i$, $i = 1, 2$, we have

$$0 = d_0 \geqslant \max\{d_1, d_2, d_3\} \geqslant \max\{\pm d_i; i = 1, 2\} \geqslant 0.$$

Hence $d_i = 0$, $i = 1, 2, 3$. In other words, $s_A(e_i) = \langle e_i, P\rangle$, $i = 1, 2$, and $\langle e, P\rangle \leqslant s_A(e)$ for $e \in (e_1, e_2)$. We prove next that $P \in A$. Suppose the contrary. Then $0 \notin A - P$. The support function of this set is $s_A(e) - \langle e, P\rangle$. Since $s_A(e_i) - \langle e_i, P\rangle = 0$ we can apply Lemma 2.3. Thus, for $e \in (e_1, e_2)$, we have the contradiction $0 > s_A(e) - \langle e, P\rangle \geqslant 0$.

Once we know that $P \in A$, we get from Lemma 2.6 and (v) that $P = M$.

Let us now consider the case $d_0 > 0$. Again set $M := M(A; e'', e')$. By Proposition 2.1(c) we know that $s_A(e) - \langle e, M\rangle > d_0$ for $e \in S\backslash[e'', e'] = (e', e'')$. Since $[e'', e'] \subset [e_1, e_2]$ we have

(vii) $s_A(e_i) - \langle e_i, M\rangle \geqslant d_0 \geqslant d_i = s_A(e_i) - \langle e_i, P\rangle$ $i = 1, 2$,

(viii) $\langle e_i, P - M\rangle \geqslant 0$ $i = 1, 2$.

Since $0 < (e_1, e_2) < \pi$ we get from here that $\langle e, P - M\rangle \geqslant 0$ for each $e \in (e_1, e_2)$. In particular, for $e^* \in [e'', e'] \subset [e_1, e_2]$ we have $\langle e^*, P\rangle \geqslant \langle e^*, M\rangle$. Then

(ix) $d_0 = \langle e^*, M\rangle - s_A(e^*) \leqslant \langle e^*, P\rangle - s_A(e^*) \leqslant d_3 \leqslant d_0$. Thus $d_0 = d_3$. Condition (ix) implies also that everywhere in (vii), (viii), and (ix) we have equalities. This is possible only if $d_0 = d_1 = d_2$, $M = P$ and $e_1 = e''$, $e_2 = e'$.

2.8. COROLLARY. *If not all of the numbers* d_1, d_2, d_3 *from Proposition* 2.7 *are equal, then* $d_0 < \max\{d_1, d_2, d_3\}$.

2.9. DEFINITION. Let Δ be a nondegenerated k-gon with vertices $M_1, M_2,..., M_k$ and side directions $e_1, e_2,..., e_k$. Then Δ is said to be alternating for $A \in \text{CONV}$ if the Hausdorff distance $h(A, \Delta)$ between A and Δ satisfies the requirements:

(a) $h(A, \Delta) = s_A(e_i) - s_\Delta(e_i)$, $i = 1, 2,..., k$;

(b) there exists $e_i^* \in (e_{i-1}, e_i)$, $i = 1, 2,..., k$ $(e_0 := e_k)$ such that $h(A, \Delta) = s_0(e_i^*) - s_A(e_i^*)$, $i = 1, 2,..., k$. The points (vectors) e_i, e_i^*, $i = 1, 2,..., k$ will be called "alternating points" of the pair (A, Δ).

3. Main Results

3.0. Corollary. *If the nondegenerated k-gon Δ with vertices $P_1, P_2, ..., P_k$ is alternating for the convex set A, then $d(P_i, A) = h(A, \Delta)$, $i = 1, 2, ..., k$, i.e., the vertices of Δ are at the same distance from A.*

Proof. This follows immediately from Corollary 2.5 and Definition 2.9 (alternating k-hon).

3.1. Theorem. *Let $A \in$ CONV have interior points, let $n \geqslant 3$, and let Δ be a best Hausdorff approximation for A in POLY_n. Then Δ is alternating for A.*

Proof. Let $k \leqslant n$ and $\Delta = (P_1, P_2, ..., P_k)$ has side directions $e_1, e_2, ..., e_k$. Let Δ be a best approximation in POLY_n for $A \in$ CONV. Put $d := h(A, \Delta)$. It is enough to prove that $d = s_A(e_i) - \langle e_i, P\rangle = \max\{\langle e, P_i\rangle - s_A(e)$: $e \in [e_{i-1}, e_i]\}$ for each $i = 1, 2, ..., k$. Suppose, for example, that at least one of the three numbers $d_1 = s_A(e_1) - \langle e_1, P_2\rangle$, $d_2 = s_A(e_2) - \langle e_2, P_2\rangle$, $d_3 = \max\{\langle e, P_2\rangle - s(e)$: $e \in [e_1, e_2]\}$ is strictly less than d. Then we will construct a k_1-gon ($k_1 \leqslant k$) Δ' with side directions $e'_1, e'_2, ..., e'_{k_1}$ such that $h(A, \Delta') \leqslant d$ and $\max\{|s_{\Delta'}(e) - s_A(e)|$: $e \in [e'_1, e'_2]\} < d$. The same argument with Δ' instead of Δ will bring us to another k_2-gon, $k_2 \leqslant k_1$, Δ'' such that $h(A, \Delta'') \leqslant d$ and $\max\{|s_{\Delta''}(e) - s_A(e)|$: $e \in [e''_1, e''_2] \cup [e''_2, e''_3]\} < d$. Proceeding in this way, after a finite number of steps we arrive at some n-gon which approximates A better than Δ, a contradiction.

Consider the points $M'_{i+1} := M(A; e_i, e_{i+1})$, $i = 1, 2, ..., k$ ($e_{k+1} := e_1$, $M'_1 := M'_{k+1}$), where e_i are the side directions of Δ, and take Δ' to be the convex hull of $\{M'_i\}_{i=1}^k$. We will show first that all points M'_i, $i = 1, 2, ..., k$ are vertices of Δ'. Those of the points M'_i which do not belong to A are necessarily different, because the corresponding vectors $e^*_{i+1} = e^*(A; e_i, e_{i+1})$ lie in different arcs (e_i, e_{i+1}). If two points M'_i, M'_j lie in A, they may coincide. But the fact $M'_i = M'_j$ has some consequences. Suppose for simplicity that $i < j$ and consider the case $(e_{i+1}, e_j) < \pi$ (the other case $(e_{j+1}, e_i) < \pi$ is treated similarly). From Lemma 2.6 we see that $s_A(e) = \langle e, M\rangle$ for $e \in (e_i, e_{j+1})$, where $M = M'_i = M'_j$. As is easily seen from the construction of M'_i, we have now $M'_i = M'_{i+1} = \cdots = M'_j = M(A; e_i, e_{j+1})$.

Identifying (if necessary) the coinciding points M'_i and introducing new indices for M'_i and e_i we may assume that M'_i, $i = 1, 2, ..., k_1$, $k_1 \leqslant k$, are different points and that the vectors $e^*_{i+1} = e^*(A; e_i, e_{i+1})$, $i = 1, 2, ..., k_1$, belong to $[e_i, e_{i+1}]$. Since A has interior points the length of the arc $\{e \in S$: $s(e) = \langle e, M\rangle\}$, where $M \in A$ is any point in A, is less than π. This means the above identification may reduce the number of points at most to $k_1 = 3$. In general $k_1 \geqslant 3$.

3.2. Lemma. *For* $i \neq j$ $\langle e_i^*, M_i' \rangle > \langle e_i^*, M_j' \rangle$.

Proof. To prove this we use the following inequalities: $\langle e_i^*, M_i' \rangle - s_A(e_i^*) = d(M_i', A) \geqslant^{(\alpha)} -d(M_j', A) \geqslant^{(\beta)} \langle e_i^*, M_j' \rangle - s_A(e_i^*)$ which, in the case $d(M_i', A) > 0$, $d(M_j', A) > 0$ are direct corollaries of Proposition 2.1 because $e_i^* \notin [e_j, e_{j+1}]$. But (α) and (β) also hold true for the case when one or both of the numbers $d(M_i', A)$, $d(M_j', A)$ are equal to 0. Evidently, the desired inequality will be derived if at least one of (α) and (β) is a strict inequality. This is the case when at least one of the numbers $d(M_i', A)$, $d(M_j', A)$ is positive. To complete the proof we have to consider also the case $d(M_i', A) = d(M_j', A) = 0$, i.e., $M_i' \in A$, $M_j' \in A$. From Lemma 2.6 we now get $s_A(e_i^*) = \langle e_i^*, M_i' \rangle > \langle e_i^*, M_j' \rangle$. The lemma is proved.

It easily implies that no three points from the set $\{M_i'\}_{i=1}^{k_i}$ belong to one straight line. Thus $\Delta' = \mathrm{co}\{M_i'\}_{i=1}^{k_1}$ is a nondegenerated k_1-gon with vertices M_i', $i = 1, 2, \ldots, k_1$, and side directions e_i', $i = 1, 2, \ldots, k_1$, determined by the conditions $\langle e_i', M_i' \rangle = \langle e_i', M_{i+1}' \rangle$, $e_i' \in [e_i^*, e_{i+1}^*]$. Let us now turn back to the proof of Theorem 3.1. It will be completed if we show that max $\{|\langle M_2', e \rangle - s_A(e)|: e \in [e_1', e_2']\} < d = h(A, \Delta)$. This will be derived from the assumption that at least one of the three numbers d_1, d_2, d_3 is strictly less than d. Since the arcs (e_1', e_2') and (e_1, e_2) have nonempty intersection (both contain e_2^*), four different situations may appear with respect to the common disposition of $[e_1, e_2]$ and $[e_1', e_2']$:

(1) $[e_1', e_2'] \subset [e_1, e_2]$,

(2) $e_1' \in [e_1, e_2]$, $e_2' \notin [e_1, e_2]$,

(3) $e_1' \notin [e_1, e_2]$, $e_2' \in [e_1, e_2]$,

(4) $[e_1', e_2'] \supset [e_1, e_2]$.

Since (see Corollary 2.8 and (iv)) $d > d_0(A; e_1, e_2) = \max\{|\langle e, M_2' \rangle - s_A(e)|: e \in [e_1, e_2]\}$, case (1) is not interesting. As will be seen from the argument below, case (4) may be reduced to cases (2) and (3). Therefore (2) and (3) are the important cases. Since these two cases are similar in nature, it is enough to consider only case (2) which is depicted in Fig. 2. We have $\langle e_2, M_2' \rangle > \langle e_2, M_3' \rangle$ and therefore (Lemma 2.2) $\langle e_1, M_2' \rangle > \langle e, M_3' \rangle$ for every $e \in (e_2, e_2')$. Now since $e_2' \in [e_2, e_3]$ we have, for $e \in (e_2, e_2')$, $s_A(e) - \langle e, M_2' \rangle < s_A(e) - \langle e, M_3' \rangle \leqslant d_0(A; e_2, e_3) \leqslant d$.

On the other hand, Proposition 2.1(c) asserts that $0 \leqslant d_0(A; e_1, e_2) \leqslant s_A(e) - \langle M_2', e \rangle$ whenever $e \in [e_2, e_2']$, i.e., $\max\{|\langle e, M_2' \rangle - s_A(e)|: e \in [e_2, e_2']\} < d$. Together with $d_0(A; e_1, e_2) < d$ this completes the proof.

3.3. Proposition. *Let* Δ *be a best approximation in* POLY_n *for* $A \in \mathrm{CONV} \backslash \mathrm{POLY}_n$. *Then* Δ *is a nondegenerated n-gon.*

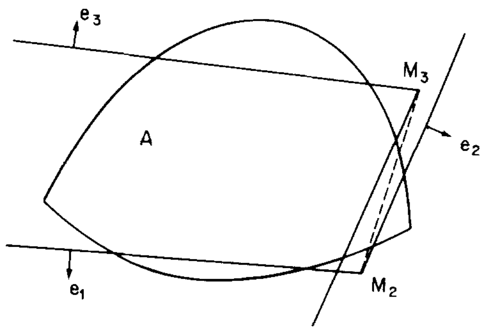

FIGURE 2

Proof. The idea is very simple. Put $\varepsilon = h(A, \Delta) > 0$ and suppose $\Delta = (P_1, P_2, ..., P_m)$, where $m < n$. Take a point P_0 belonging to the ε neighbourhood of the set $A \subset P^2$. The new $(m+1)$-gon $\Delta' = (P_0, P_1, ..., P_m)$ belongs to POLY_n and $h(A, \Delta') = h(A, \Delta)$. Therefore Δ' is a best approximation for A in POLY_n. On the other hand P_0 may be chosen in such a way that Δ' be nonalternating for A. This contradicts Theorem 3.1 and completes the proof.

Another application of the alternating property is the following result which was also observed by N. Živkov.

3.4. PROPOSITION. *Let Δ be a best approximation for A in* POLY_n. *Then the set $A_t = tA + (1-t)\Delta$, $0 < t < 1$, has unique best approximation in* POLY_n *and this best approximation is Δ.*

Proof. Denote by $s_1(e)$, $s_0(e)$, and $s_t(e)$ the support function of the sets A, Δ, and A_t, respectively. Evidently, $s_t = ts_1 + (1-t)\, s_0$ and therefore

(x) $s_t - s_0 = t(s_1 - s_0)$,

(xi) $t(s_t - s_1) = (1-t)(s_0 - s_t)$.

From (x) we see that the alternation points of (A_t, Δ) are alternating for (A, Δ) and vice versa. Freom (xi) it also follows that the alternating points of (A_t, Δ) are just those points where the function $s_t(e) - s_1(e)$ attains its maximal (minimal) possible values.

Without loss of generality we may assume that $\|s_1 - s_0\| = 1$. Then $\|s_t - s_0\| = t$ and $\|s_t - s_1\| = 1 - t$. We show first that Δ is a best approximation in POLY_n for A_t. Indeed, if there exists some $\Delta' \in \text{POLY}_n$ with $\|s_t - s_{\Delta'}\| < t$ we would get the contradiction $\|s_1 - s_{\Delta'}\| \leqslant \|s_1 - s_t\| + \|s_t - s_{\Delta'}\| < (1-t) + t = 1$. Next we show that Δ is the only best approximation of A_t. To do this we consider one arbitrary best approximation Δ' of

A_t (i.e., $\|s_t - s_{\Delta'}\| = t$) and show that Δ' is a best approximation for A and that the pair (A, Δ') has the same alternating points as the pair (A, Δ). This will be enough to conclude that $\Delta = \Delta'$. From $1 \leqslant \|s_1 - s_{\Delta'}\| \leqslant \|s_1 - s_t\| + \|s_t - s_{\Delta'}\| = (1-t) + t = 1$ we see that Δ' is a best approximation for A.

Let e_0 be an alternating point for (A, Δ'). For example, $1 = s_1(e_0) - s_{\Delta'}(e_0)$. Then $1 = (s_1(e_0) - s_t(e_0)) + (s_t(e_0) - s_{\Delta'}(e_0)) \leqslant (1-t) + t = 1$. Hence $s_1(e_0) - s_t(e_0) = 1 - t$ and $s_t(e_0) - s_{\Delta'}(e_0) = t$. This means e_0 is an alternating point for the pair (A, A_t). By (x) and (xi) e_0 will be alternating for the pair (A, Δ). Similarly, if $-1 = s_1(e^*) - s_{\Delta'}(e^*)$ we have $-1 = (s_1(e^*) - s_t(e^*)) + (s_t(e^*) - s_{\Delta'}(e^*)) \geqslant -(1-t) - t = -1$. As above, again using (x) and (xi), we see that e^* is alternating for A and A_t and therefore for A and Δ. By Proposition 3.3 the set of alternating points of (A, Δ') contains all alternating points of (A, Δ).

We are now in a position to prove

3.5. THEOREM. *The set of all those* $A \in$ CONV *which have unique best approximation in* POLY$_n$ *for every* $n \geqslant 3$, *contains a dense* G_δ *subset of* (CONV, h). *That is, the set* $\{A \in$ CONV: *A has more than one best approximation in at least one* POLY$_n$, $n \geqslant 3\}$ *is of the first Baire category in* (CONV, h).

Proof. One way to prove this assertion is given in Gruber and Kenderov [5]. In Kenderov [11] another way was outlined. Here we suggest an argument which is based on Proposition 3.4.

Fix $n = k$ and consider the metric projection π_k: CONV $\to$ POLY$_k$ assigning to each $A \in$ CONV the set $\pi_k(A)$ of all best approximations for A in POLY$_k$. By the Blaschke selection theorem POLY$_k$ is an approximatively compact subset of $C(S)$. The result of *I.* Singer [16] asserts that the metric projection π_k: (CONV, h) $\to$ (POLY$_k$, h) is an upper semicontinuous set-valued map with compact images. According to a theorem of Fort [2] there exists a dense G_δ subset W_k of (CONV, h) at the elements of which π_k is lower semicontinuous, i.e., for every $A \in W_k$, $\varepsilon > 0$, and $\Delta \in \pi_k(A)$ there exists $\delta > 0$ such that for every $A' \in$ CONV, $h(A, A') < \delta$, there exists $\Delta' \in \pi_k(A')$ for which $h(\Delta, \Delta') < \varepsilon$. We will show now that every $A \in W_k$ has unique best approximation in POLY$_k$. Take such an $A \in W_k$ and suppose there exist $\Delta_1, \Delta_2 \in \pi_k(A)$, $\Delta_1 \neq \Delta_2$. Put $\varepsilon = \frac{1}{2} h(\Delta_1, \Delta_2) > 0$ and consider the set $A_t = tA + (1-t)\Delta_1$. According to Proposition 3.4 $\pi_k(A) = \{\Delta_1\}$ for every $t > 0$. As $\lim_{t \to 0} h(A_t, A) = 0$ this contradicts the lower semicontinuity of π_k at A, because $h(\Delta_1, \Delta_2) > \varepsilon$. The theorem is proved because $\bigcap_{k=3}^{\infty} W_k$ is again a dense G_δ subset of CONV.

3.6. *Remark.* This theorem goes along the line started in the papers of Stechkin [17] and Garkavi [3, 4]. Results about the uniqueness of the best approximations for "almost all" elements of the space are contained in the papers by Konijagin [13, 14], Zajičik [19], Živkov [20, 21] and Kenderov [7–10]. It does not seem that Theorem 3.5 is a corollary of the results from these papers because neither $C(S)$ (in the "sup" norm) is strictly convex space, nor is the structure of the set POLY_n simple (it is not a convex subset of CONV).

4. Best Approximation with a Fixed Side Direction

We discuss here another approximation problem in which the best approximation is obliged to have one of its side directions coinciding with a given vector $e \in S$. It turns out (under reasonable restrictions) that this problem always has a solution and this solution is unique. It will also be shown that for every $A \in \mathrm{CONV}$ there are a lot of alternating n-gons, $n \geqslant 3$. A necessary and sufficient condition will be given for some $A \in \mathrm{CONV}$ to be an n-gon.

First we need some constructions.

4.0. Construction. Let $A \in \mathrm{CONV}$. For $e \in S$ we set $w(e) = s_A(e) + s_A(-e)$ and recall that this is the "width of A in direction e." As $\operatorname{int} A \neq \varnothing$, $w(e) > 0$ for every $e \in S$. To each $e \in S$ and a real number d, $0 < d \leqslant \frac{1}{2} w(e)$, we put into correspondence a point $M = M(A; e, d)$ and e vector $e^* = e^*(A; e, d)$ such that

(1) $d = d(M, A) = \langle e^*, M \rangle - s_A(e^*)$,

(2) $s_A(e) - \langle e, M \rangle = d$,

(3) $e^* \in (e, -e)$.

First consider the line $L = \{X \in R^2 \colon \langle X, e \rangle = s_A(e) - d\}$. Because of the condition $0 < d \leqslant \frac{1}{2} w(e)$, L intersects A and therefore will intersect the interior of $A + dB$ (this set is the d neighbourhood of A). Then L crosses the boundary of $A + dB$ at two points M_1 and M_2 which are different. Both M_1 and M_2 satisfy (2). Denote by e_i^*, $i = 1, 2$, the unit vectors uniquely determined by the condition $d = d(M, A) = \langle e_i^*, M_i \rangle - S_A(e_i^*)$ (i.e., each of e_1^* and e_2^* satisfies (1)). Now we prove that each of the arcs $(e, -e)$, $(-e, e)$ contains only one of the vectors e_i^*, $i = 1, 2$. Using (1) and (2) it is not difficult to see that $e_i^* \neq \pm e$ $i = 1, 2$. Indeed, suppose $e_i^* = e$. Then $s_A(e) = s_A(e_i^*) = \langle e_i^*, M_i \rangle - d = \langle e, M_i \rangle - d = s_A(e) - 2d$. As $d > 0$ this is a contradiction. Analogously we disprove the relation $e_i^* = -e$: $0 < w(e) = s_A(e) + s_A(-e) = s_A(e) + s_A(e_i^*) = s_A(e) + \langle e_i^*, M_i \rangle - d = s_A(e) + \langle -e, M_i \rangle - d = s_A(e) - s_A(e) = 0$.

Now we prove that each of the arcs $(e, -e)$, $(-e, e)$ contains only one of the vectors e_i^*, $i=1,2$. Consider $\langle e_2^*, M_2 - M_1\rangle = \langle e_2^*, M_2\rangle - \langle e_2^*, M_1\rangle = s_A(e_2^*) + d - \langle e_2^*, M_1\rangle \geqslant d - \max\{(\langle e, M_1\rangle - s_A(e)) : e \in S\} = d - d = 0$. As $e_2^* \neq \pm e$ we get from here $\langle e_2^*, M_2 - M_1\rangle > 0$.

Similarly we derive $\langle e_1^*, M_1 - M_2\rangle > 0$. These inequalities imply that each of the arcs contain only one of e_i^*, $i=1,2$. In what follows we will denote by $e^* = e^*(A; e, d)$ that vector e_i^* which belongs to the arc $(e, -e)$. The corresponding point M_i will be denoted by $M = M(A; e, d)$. Evidently (1)–(3) are satisfied. It is now clear that these three conditions completely determine e^* and M. Moreover, the above argument shows that (3) can be replaced by the (formally less restrictive) condition

(3′) $e^* \in [e, -e]$.

4.1. LEMMA. *The defined mappings* $(e,d) \to M(A; e, d)$ *and* $(e, d) \to e^*(A; e, d)$ *are continuous at every point* (e_0, d_0), $e_0 \in S$, $d_0 > 0$.

Proof. The argument follows the scheme by means of which continuity of an implicitely defined function is proved. Let $e_i \to e_0$, $d_i \to d_0$, where $e_i \in S$, $0 < d_i \leqslant \frac{1}{2}w(e_i)$, $i = 0, 1, 2, \ldots$. Set $e_i^* = e^*(A; e_i, d_i)$ and $M_i = M(A; e_i, d_i)$ $i = 0, 1, 2, \ldots$. Then

(1_i) $d_i = d(M_i, A) = \langle e_i^*, M_i\rangle - s_A(e_i^*)$,

(2_i) $d_i = s_A(e_i) - \langle e_i, M_i\rangle$,

(3_i) $e_i^* \in (e_i, -e_i)$.

Since all M_i belong to a bounded subset of R^2 there will exist a converging subsequence. The situation with $\{e_i^*\}_{i \geqslant 1} \subset S$ is analogous. For simplicity we assume that $\{M_i\}_i$ tends to some M and $\{e_i^*\}_i$ converges to some $e^* \in S$. Taking limits in (1_i)–(3_i) we get

(1) $d_0 = d(M, A) = \langle e^*, M\rangle - s_A(e^*)$,

(2) $d_0 = s_A(e_0) - \langle e_0, M\rangle$,

(3′) $e^* \in [e_0, -e_0]$.

By the construction, these three conditions imply $M = M_0$, $e^* = e_0^*$.

Taking Proposition 2.1 into account we see that, to every point $M \notin A$, there correspond two vectors e^*, e' determined by the conditions

(a) $d(M, A) = s_A(e') - \langle e', M\rangle$,

(b) $d(M, A) = \langle e^*, M\rangle - s_A(e^*)$,

(c) $e' \in (e^*, -e^*)$.

Proceeding like in the previous result we can prove that thus defined e^* and e' depend continuous on M. Hence the composition mapping assigning to each pair (e, d), $e \in S$, $0 < d \leqslant \frac{1}{2}w(e)$ the vector e' (via the maps $(e, d) \mapsto$

$M \mapsto e'$) is also continuous. We will denote e' by $T(e, d)$. The next result is now evident.

4.2. PROPOSITION. *The mapping* $(e, d) \mapsto T(e, d)$ *is continuous at every point* (e, d), *where* $0 < d \leqslant \frac{1}{2}w(e)$.

Let us now calculate $T(e, d)$ for $d = \frac{1}{2}w(e)$. From $s_A(-e) - \langle -e, M\rangle = s_A(-e) + \langle e, M\rangle = s_A(-e) + s_A(e) - d = w(e) - d = d$ we see that $T(e, \frac{1}{2}w(e)) = -e$, i.e., $(e, T(e, \frac{1}{2}w(e))) = \pi$. If $d < \frac{1}{2}w(e)$, the same argument gives $s_A(-e) - \langle -e, M\rangle > d$. Combined with Proposition 2.1 this leads to the conclusions $e^* \in (e, -e)$, $e' \in (e^*, -e)$, i.e., $e' = T(e, d) \in (e, -e)$.

Further we need one more definition. For every $e \in S$, positive integer k and a real number d, $0 < d \leqslant \frac{1}{2}w(e)$, we define inductively $T^k(e, d)$. $T^1(e, d) = T(e, d)$ and $T^{k+1}(e, d) = T(T^k(e, d), d)$. The correctness of this definition is based on the fact that $w(T^k(e, d)) \geqslant 2d$ whenever $w(e) \geqslant 2d$.

4.3. LEMMA. *Let* $0 < d \leqslant \frac{1}{2}w(e)$. *Then* $w(T(e, d)) \geqslant 2d$.

Proof. If $d = \frac{1}{2}w(e)$, $T(e, d) = -e$, then the lemma follows from $w(e) = w(-e)$. Let us consider the case $d < \frac{1}{2}w(e)$. Since $e' = T(e, d) \in (e, -e)$, we have $-e' \notin [e, e']$. By Proposition 2.1 $d < s_A(-e') - \langle -e', M\rangle = s_A(-e') + \langle e', M\rangle = s_A(-e') + s_A(e') - d = w(e') - d$. Lemma 4.3 is proved.

Evidently, the mapping $T^k(e, d)$ is continuous. The real-valued function $f^k(e, d)$ defined inductively by $f^1(e, d) = (e, T(e, d))$, $f^{k+1}(e, d) = f^k(e, d) + f^1(T^k(e, d), d) = f^k(e, d) + (T^k(e, d), T^{k+1}(e, d))$ will be continuous. Clearly $f^k(e, \frac{1}{2}w(e)) = k\pi$.

4.4. COROLLARY. *Let* $e \in S$. *In the interval* $(0, \frac{1}{2}w(e)]$ $f^k(e, d)$ *is strictly increasing as a function of* d.

Proof. We want to prove that from $\frac{1}{2}w(e) > d_1 > d_2 > 0$ it follows $f^k(e, d_1) > f^k(e, d_2)$. This will be done by induction. A direct application of Proposition 2.7 shows that the arc $[e, T(e, d_1)]$ is not contained in $[e, T(e, d_2)]$. Thus, for $k = 1$, the problem is settled. Suppose the assertion is true for $f^k(e, d)$: $f^k(e, d_1) > f^k(e, d_2)$. We prove the same inequality for f^{k+1}. There is sense to consider only the case when $f^k(e, d_1) \leqslant f^{k+1}(e, d_2)$ (otherwise the required inequality follows from $f^{k+1}(e, d_1) > f^k(e, d_1)$). In other words, $f^k(e, d_2) < f^k(e, d_1) \leqslant f^{k+1}(e, d_2)$. This corresponds to the case when $T^k(e, d_1) \in (T^k(e, d_2), T^{k+1}(e, d_2)] = (T^k(e, d_2), T(T^k(e, d_2), d_2)]$. That $T^{k+1}(e, d_1)$ does not belong to this arc is again a corollary of Proposition 2.7.

For convenience we denote by $f^k(e, 0)$, $\lim_{d \to 0} f^k(e, d)$ and by $T(e, 0)$ such a vector from S that $f^1(e, 0) = (e, T(e, 0))$. It is clear what $T^k(e, 0)$ means.

The function $f^k(e, 0)$ is a convenient tool to express the fact that a given set A is an n-gon.

4.5. THEOREM. *Let* $e \in S$ *and* $A \in \mathrm{CONV}$, $\mathrm{int}\, A \neq \emptyset$. *The set* A *is a nondegenerated* n*-gon with* e *among its side directions if and only if* $f^n(e, 0) = 2\pi$.

The proof will need several auxiliary results.

4.6. LEMMA. *Let* $M \in A$ *and* $s_A(e) = \langle e, M\rangle$ *for* $e \in (e_0, e_0')$. *Then*

(1) $e_0' \in [e, T(e, 0)]$ *for each* $e \in [e_0, e_0']$.

(2) $s_A(e) = \langle e, M\rangle$, *whenever* $e \in [e_0, T(e_0, 0)]$, *and* $s_A(e) > \langle e, M\rangle$ *for* $e \in (T(e_0, 0), -e_0)$.

(3) $T(e, 0) = T(e_0, 0)$ *for each* $e \in [e_0, T(e_0, 0))$.

Proof. (1) Take $d > 0$ and $e \in [e_0, e_0']$. From Proposition 2.7 (with $e'' := e$, $e' := T(e, d)$, $e_1 := e_0$, and $e_2 := e_0'$) we see that the arc $(e, T(e, d))$ cannot be contained in (e_0, e_0'). Therefore $e_0' \in (e, T(e, d))$ for every $d > 0$. Thus $e_0' \in [e, T(e, 0)]$.

(2) Take a sequence $\{d_j\}_{j\geqslant 1}$ of positive real numbers, $\lim_j d_j = 0$. Then the sequence $\{e_j' = T(e_0, d_j)\}_{j\geqslant 1}$ converges to $T(e_0, 0)$ and $\{M_j = M(A; e_0, d_j)\}$ contains a converging (to some point $M \in R^2$), subsequence. Taking limits in $d_j = d(M_j, A) = s_A(e_0) - \langle e_0, M_j\rangle = s_A(e_j') - \langle e_j', M_j\rangle$, we obtain $M \in A$, $s_A(e_0) = \langle e_0, M\rangle$ and $s_A(T(e_0, 0)) = \langle T(e_0, 0), M\rangle$. By Lemma 2.6 $s_A(e) = \langle e, M\rangle$ for every $e \in [e_0, T(e_0, 0)]$. From part (1) and 2.6 it is also seen that $s_A(e_1) \neq \langle e_1, M\rangle$ for any $e_1 \in (T(e_0, 0), -e_0)$. Therefore $s_A(e_1) > \langle e_1, M\rangle$.

(3) By (1) it follows that $T(e_0, 0) \subset [e, T(e,0)]$. Since (by the proof of (2)) $s_A(T(e, 0)) = \langle T(e, 0), M\rangle$, from (2) we get $T(e, 0) \in [e_0, T(e_0, 0)]$.

4.7. LEMMA. *Let the vectors* $e_0, e_j \in S$, $j = 1, 2, 3, \ldots$, *and the positive real numbers* d_j, $j = 1, 2, 3, \ldots$, *be such that the sequences* $\{|e_0, e_j|\}_{j\geqslant 1}$, $\{d_j\}_{j\geqslant 1}$ *decrease to* 0. *Suppose* $t := \limsup_j f^1(e_j, d_j) > 0$. *Then there exist* $M \in R^2$ *and* $e_0' \in S$ *such that*

(1) $(e_0, e_0') = t$,

(2) $s_A(e) = \langle e, M\rangle$ *whenever* $e \in (e_0, e_0')$.

Proof. Set $e_j' = T(e_j, d_j)$ and $M_j = M(A; e_j, d_j)$. Then $d_j = d(M_j, A)$ and $d_j = s_A(e_j) - \langle e_j, M_j\rangle = s_A(e_j') - \langle e_j', M_j\rangle$. Without loss of generality we may assume that $\{M_j\}_{j\geqslant 1}$, $\{e_j'\}_{j\geqslant 1}$ and $\{f^1(e_j, d_j)\}_{j\geqslant 1}$ are convergent sequences. Taking limits we get $d(M, A) = 0$, $s_A(e_0) = \langle e_0, M\rangle$ and $s_A(e_0') = \langle e_0', M\rangle$, where $M = \lim_j M_j$, $e_0' = \lim e_j'$. From the condition $(e_j, e_j') = f^1(e_j, d_j) \leqslant \pi$ we see that $\pi \geqslant (e_0, e_0') = t > 0$.

It remains to apply Lemma 2.6 in order to complete the proof. However, we must prove first that $t \neq \pi$. Here is one possible way to do this. Since A contains a circle with radius $r_0 > 0$, $w(e) \geqslant 2r_0$ for every $e \in S$. Therefore, when $0 < d_0 < r_0$, $f^1(e_0, d_0) < \pi$ for each $e \in S$. Since the function $f^1(\cdot, d_0)$ is continuous and S is compact $u := \max\{f^1(e, d_0)\colon e \in S\} < \pi$. When $d_j < d_0, f'(e_j, d_j) < f^1(e_j, d_0) \leqslant u < \pi$. Thus $t = \lim_j f^1(e_j, d_j) \leqslant u < \pi$.

4.8. COROLLARY. *Let* e_0, e_j, d_j $j = 1, 2, \ldots,$ *be as in Lemma* 4.7. *Then* $f^1(e_0, 0) = \lim_j f^1(e_0, d_j) = \lim_j f^1(e_j, d_j)$.

Proof. From Proposition 2.7 we derive

$$[e_0, T(e_0, d_j)] \subset [e_0, e_j] \cup [e_j, T(e_j, d_j)] = [e_0, T(e_j, d_j)].$$

Therefore $f^1(e_0, 0) \leqslant \liminf_j f^1(e_j, d_j) \leqslant \limsup_j f^1(e_j, d_j) =: t$. It remains to prove that $t \leqslant f^1(e_0, 0)$. If $t = 0$, there is nothing to prove since $f^1(e, 0) \geqslant 0$. If $t > 0$, the inequality $f^1(e_0, 0) \geqslant t$ is a corollary of Lemmas 4.6 and 4.7.

4.9. Corollary. *Let* $e \in S$, $A \in \mathrm{CONV}$ *and* $e_i = T^i(e, 0)$ $i = 1, 2, \ldots,$ $k-1$. *Then* $f^k(e, 0) = f^1(e, 0) + f^1(e_1, 0) + \cdots + f^1(e_{k-1}, 0)$.

Proof. Let $k = 2$. Take a decreasing sequence $\{d_j\}_{j \geqslant 1}$ of real numbers, $\lim_j d_j = 0$. Then $f^2(e, 0) = \lim_j f^2(e, d_j) = \lim_j (f^1(e, d_j) + f^1(e_j, d_j))$, where $e_j = T(e, d_j)$. We know that $\{e_j\}_{j \geqslant 1}$ converges to $e_1 = T(e, 0)$ in such a way that $[e_1, e_j]$ decreases to 0. From Corollary 4.8 it follows that $f^2(e, 0) = f^1(e, 0) + f^1(e_1, 0)$. Analogously we proceed when $k \geqslant 3$.

Let us now turn back to the proof of theorem (4.5). Let A be a nondegenerated n-gon with side directions $e_1, e_2, \ldots, e_{n-1}, e_n$, where $e_1 = e$. By what was proved in Corollary 4.9 $e_{i+1} = T^i(e_i, 0)$, $i = 1, 2, \ldots, n$. Therefore $f^n(e, 0) = (e_1, e_2) + (e_2, e_3) + \cdots + (e_k, e_1) = 2\pi$.

Now let A be such a convex set that $f^n(e, 0) = 2\pi$. It is not difficult to understand that $0 < f^1(e, 0) < f^2(e, 0) < \cdots < f^n(e, 0) = 2\pi$. Put $e_1 = e$, $e_2 = T^1(e, 0), \ldots, e_k = T^{k-1}(e, 0)$. Evidently $e_1 = T(e_k, 0)$. By Lemma 4.7 for every arc (e_i, e_{i+1}) there is a point P_i for which $s_A(e) = \langle e, P_i \rangle$ when $e \in [e_i, e_{i+1}]$. Therefore A is a nondegenerated n-gon.

4.10. THEOREM. *Let the vector* $e \in S$ *and the set* $A \in \mathrm{CONV}$, $\mathrm{int}\, A \neq \varnothing$ *be such that* $f^n(e, 0) \leqslant 2\pi$, *where* $n \geqslant 3$. *Then, for each positive integer* k, $3 \leqslant k \leqslant n$, *there exists just one nondegenerated* k*-gon* Δ *which is alternating for* A *and has* e *among its side directions.*

Proof. We consider two subcases

(a) $f^n(e, 0) < 2\pi$,

(b) $f^n(e, 0) = 2\pi$.

Let us consider case (a). If $f^k(e, d) = 2\pi$ for some $d > 0$, then it is easy to realize that there exists an alternating k-gon Δ for A with side directions e, $T(e, d)$, $T^2(e, d)$,..., $T^{k-1}(e, d)$ and such that $h(A, \Delta) = d$. Conversely, if some k-gon Δ is alternating for A and e is among its side directions, then $f^k(e, d) = 2\pi$, where $d = h(A, \Delta)$. Therefore the proof will be completed with the proof of the following fact:

4.11. LEMMA. *Let $f^n(e, 0) < 2\pi$. Then for every integer k, $3 \leqslant k \leqslant n$, there exists a unique number d_k, $0 < d_k < \frac{1}{2}w(e)$, for which $f^k(e, d_k) = 2\pi$.*

Proof. Let $k = 3$. From $f^3(e, \frac{1}{2}w(e)) = 3\pi$ and $f^3(e, 0) \leqslant f^n(e, 0) < 2\pi$ it follows that there exists d_3, $0 < d_3 < \frac{1}{2}w(e)$, for which $f^3(e, d_3) = 2\pi$. This number is uniquely determined by the monotonicity of $f^3(e, \cdot)$. From $f^4(e, d_3) > f^3(e, d_3) = 2\pi$ and $f^4(e, 0) \leqslant f^n(e, 0) < 2\pi$ we derive the existence of some d_4, $0 < d_4 < d_3$, such that $f^4(e, d_4) = 2\pi$. In this way, step by step, we determine the numbers $d_3, d_4, \ldots, d_n$ so that $0 < d_n < d_{n-1} < \cdots < d_3 < \frac{1}{2}w(e)$ and $f^k(e, d_k) = 2\pi$. Case (a) is completed.

(b) In this case, according to Theorem 4.5, A is an n-gon with e among its side directions. This n-gon is alternating for itself. Since for k, $3 \leqslant k \leqslant n - 1$, $f^k(e, 0) < f^n(e, 0) = 2\pi$ the rest of the proof is contained in (a).

4.12. THEOREM. *Let $e \in S$, $A \in \mathrm{CONV}$, $\operatorname{int} A \neq \emptyset$, be such that $f^n(e, 0) \leqslant 2\pi$. Among all n-gons having e as side direction the alternating n-gon for A approximates A (in the Hausdorff metric) in the best possible way.*

Proof. The case $f^n(e, 0) = 2\pi$ is not interesting, because A is an n-gon with e among its side directions. Suppose $f^n(e, 0) < 2\pi$ and take some n-gon Δ with side directions $e_1, e_2, \ldots, e_n$, $e_1 = e$. Then $d = h(A, \Delta) > 0$. From Proposition 2.7 we have that $f^1(e, d) \geqslant [e_1, e_2]$ and that the inequality is strict if $T(e, d) \neq e_2$. Similarly, $f^2(e, d) \geqslant [e_1, e_2] + [e_2, e_3]$ and the inequality is again strict if one of the conditions $T^i(e, d) = e_{i+1}$, $i = 1, 2$, is violated. Repeating this argument we arrive at the inequality $f^n(e, d) \geqslant [e_1, e_2] + \cdots + [e_n, e_1] = 2\pi$ which is strict if $T^i(e, d) \neq e_{i+1}$ for some $i = 1, 2, \ldots, n$. If Δ is not alternating, then $f^n(e, d) > 2\pi$ and there exists a number $d^* < d$ for which $f^n(e, d^*) = 2\pi$. The latter condition implies the existence of some n-gon Δ^* which is alternating for A, has e among its side directions and $h(A, \Delta^*) = d^* < d$. The theorem is proved.

REFERENCES

1. D. E. McCLURE AND R. A. VITALE, Polygonal approximation of plane convex bodies, *J. Math. Anal. Appl.* **51** (1975), 326–358.
2. M. K. FORT, Points of continuity of semicontinuous functions, *Publ. Math. Debrecen* 2 (1951). 100–102.
3. A. L. GARKAVI, Almost Čebyšev systems of continuous functions, *Izv. Vysš. Učebn. Zaved. Matematika* **45** (1965), 36–44; translation, *Amer. Math. Soc. Transl.* **96** (1970), 177–187.
4. A. L. GARKAVI, On Čebyšev and almost Čebyšev subspaces, *Izv. Akad. Nauk SSSR Ser. Matematika* **28** (1964), 799–818; Translation, *Amer. Math. Soc. Transl.* **96** (1970), 153–175.
5. P. GRUBER AND P. KENDEROV, Approximation of convex bodies by polytopes, *Rend. Circ. Mat. Palermo* (2) **31** (1982), 195–225.
6. R. IVANOV, Approximation of Convex n-gons by Inscribed $(n-1)$-gons, *in* "Proceedings of the Second Spring Conference of the Bulg. Math. Soc., Vidin, 1973," pp. 115–121. [Bulgarian]
7. P. S. KENDEROV, Points of Continuity of Semi-continuous Multivalued Mappings and an Application to the Theory of Metric Projections, *in* "Proceedings of the 4th Spring Conference of the Bulg. Math. Soc., Pernik, 1975," pp. 191–197.
8. P. KENDEROV, Points of single-valuedness of multivalued metric projections, *C. R. Acad. Bulgare Sci.* **29** (1976), 773–775.
9. P. S. KENDEROV, Uniqueness on a residual part of the best approximations in Banach spaces, *PLISKA Stud. Math. Bulgar.* **1** (1977), 122–127.
10. P. S. KENDEROV, Strong continuity of pointwise continuous mappings, *Pacific J. Math.* **89** (1980), 111–130.
11. P. S. KENDEROV, Approximation of plane convex compacta by polygons, C. R. *Acad. Bulgare Sci.* **33** (1980), 889–891.
12. P. S. KENDEROV, On an optimal property of the square, *C. R. Acad. Bulgare Sci.* **26** (1973), 1143–1146. [Russian]
13. S. V. KONIJAGIN, Approximative properties of arbitrary subsets of Banach spaces, *Dokl. Akad. Nauk SSSR* **239** (1978), 261–264. [Russian]
14. S. V. KONIJAGIN, On the approximative properties of closed subsets of Banach spaces and characterization of strongly convex spaces, *Dokl. Akad. Nauk SSSR* **251** (1980), 276–280.
15. V. POPOV, Approximation of convex sets, *Bull. Inst. Math. Bulgar. Acad. Sci.* **11** (1970), 67–79. [Bulgarian]
16. I. SINGER, Some remarks on approximative compactness, *Rev. Roumaine Math. Pures Appl.* **9** (1964), 167–177.
17. S. B. STECHKIN, Approximative properties of sets in linear normed spaces, *Rev. Roumaine Math. Pures Appl.* **8** (1963), 5–18. [Russian]
18. F. TÓTH, Approximation by polygons and polyhedra, *Bull. Amer. Math. Soc.* **54** (1948), 431–438.
19. L. ZAJIČEK, On the points of multivaluedness of metric projections in separable Banach spaces, *Comment. Math. Univ. Carolin.* **19** (1978), 513–523.
20. N. ŽIVKOV, Metric projections and antiprojections in strictly convex normed spaces, *C. R. Acad. Bulgare Sci.* **31** (1978), 369–371.
21. N. V. ŽIVKOV, "Continuity and Nonmultivaluedness Properties of Metric Projections and Antiprojections," Serdika, *Bulg. Math. Publ.* **8** 4 (1982), 378–385.